Cobb-Douglas profit maximization and cost minimization (short and long run)

Short-run

Let us consider a firm using capital (K) and labor (L) to produce output Y. Let's also assume the firm's production function has a Cobb-Douglas form:

$$Y = AK^{\alpha}L^{\beta} \tag{1}$$

Where A is total factor productivity, α and β are output elasticities of capital and labor respectively. In the short-run, capital (K) is fixed and labor (L) is the variable input.

Assuming the firm is a price-taker (it cannot influence the price of its product), it maximizes profit by choosing the amount of labor to employ. The firm's profit function is:

$$\pi = PY - wL - rK \tag{2}$$

Where P is the price of output and w is the wage rate. Substituting the production function into the profit function gives:

$$\pi = PAK^{\alpha}L^{\beta} - wL \tag{3}$$

The first-order condition for profit maximization with respect to labor (L) is:

$$\frac{\partial \pi}{\partial L} = \beta PAK^{\alpha}L^{\beta-1} - w = 0 \tag{4}$$

Now, let's isolate L on one side of the equation. First, we can add w to both sides and then divide by $\beta PAK^{\alpha}L^{\beta-1}$ (Assuming it is different from 0):

$$L^{\beta-1} = \frac{w}{\beta PAK^{\alpha}} \tag{5}$$

Next, to solve for L, we raise both sides to the power of $\frac{1}{\beta-1}$:

$$L = \left(\frac{w}{\beta PAK^{\alpha}}\right)^{\frac{1}{\beta-1}} \tag{6}$$

Therefore, the amount of labor that maximizes profit in the short-run, given the wage rate w, the price of output P, and the fixed capital K, is:

$$L^{*} = \left(\frac{w}{\beta PAK^{\alpha}}\right)^{\frac{1}{\beta-1}} \tag{7}$$

Long-run

For long-run profit maximization, we consider both labor (L) and capital (K) as variables. We'll continue using the Cobb-Douglas production function:

$$Y = AK^{\alpha}L^{\beta} \tag{8}$$

Assuming the firm's cost function is $C = wL + rK$, where w is the wage rate, L is labor, r is the rental rate of capital, and K is capital. The profit function $\pi = PY - C$ is then:

$$\pi = PAK^{\alpha}L^{\beta} - (wL + rK) \tag{9}$$

The first-order conditions for profit maximization with respect to labor and capital are respectively:

$$\frac{\partial \pi}{\partial L} = \beta PAK^{\alpha}L^{\beta-1} - w = 0 \tag{10}$$

$$\frac{\partial \pi}{\partial K} = \alpha PAK^{\alpha-1}L^{\beta} - r = 0 \tag{11}$$

Divide these two equations to eliminate P and A and obtain the optimal ratio of K to L:

$$\frac{K}{L} = \frac{\alpha}{\beta}\frac{w}{r} \tag{12}$$

This equation indicates the optimal capital-labor ratio for long-run profit maximization. We can express K as a function of L:

$$K = \frac{\alpha}{\beta}\frac{w}{r}L \tag{13}$$

Substituting into the first condition gives:

$$\frac{\partial \pi}{\partial L} = \beta PA\left(\frac{\alpha}{\beta}\frac{w}{r}L\right)^{\alpha} L^{\beta-1} - w = 0 \tag{14}$$

Solving for L:

$$L^{\alpha+\beta-1} = \frac{w}{\left(\frac{\alpha}{\beta}\frac{w}{r}\right)^{\alpha}\beta PA} \tag{15}$$

$$L = \left[\frac{w}{\left(\frac{\alpha}{\beta}\frac{w}{r}\right)^{\alpha}\beta PA}\right]^{\frac{1}{1-\alpha-\beta}} \tag{16}$$

Solving for K:

$$K = \frac{\alpha}{\beta}\frac{w}{r}\left[\frac{w}{\left(\frac{\alpha}{\beta}\frac{w}{r}\right)^{\alpha}\beta PA}\right]^{\frac{1}{1-\alpha-\beta}} \tag{17}$$

The solutions for optimal L and K depend on whether $1 - \alpha - \beta$ is greater than, equal to, or less than zero. Let's consider these three cases:

1. $1 - \alpha - \beta > 0$ (decreasing returns to scale): The expression $1/(1 - \alpha - \beta)$ is positive. Given that w, r, A, P, α, and β are all positive, the solutions for L and K are also positive. Therefore, we have a unique, finite solution for L and K.

2. $1 - \alpha - \beta = 0$ (constant returns to scale): The expression $1/(1 - \alpha - \beta)$ is undefined because we are dividing by zero. Therefore, the solutions for L and K are indeterminate, and we can say that there is no unique solution in this case. Note that under constant returns to scale, the firm can scale up or scale down its use of labor and capital without affecting the ratio of output to input.

3. (1-$\alpha - \beta < 0$) (increasing returns to scale). An increase in either input would lead to a more than proportional increase in output. Hence, the firm would always have an incentive to increase the use of both inputs to increase output, given that it can afford to do so. Check second-order conditions to see why maximization fails in this case.

Cost minimization in the short run

In the short run, we consider capital (K) to be fixed and labor (L) to be variable. Given a Cobb-Douglas production function:

$$Y = AK^{\alpha}L^{\beta} \tag{18}$$

And the cost of production:

$$C = wL + rK \tag{19}$$

We wish to minimize C subject to the production function, which is a constrained optimization problem. We can set this up using a Lagrangian:

$$\mathcal{L} = wL + rK + \lambda \left(Y - AK^{\alpha}L^{\beta}\right) \tag{20}$$

The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial L} = w - \lambda\beta AK^{\alpha}L^{\beta-1} = 0 \tag{21}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = Y - AK^{\alpha}L^{\beta} = 0 \tag{22}$$

We can rewrite the second equation as:

$$L = \left(\frac{Y}{AK^{\alpha}}\right)^{\frac{1}{\beta}} \tag{23}$$

So the optimal level of labor (L^*) that minimizes cost in the short run, given a fixed level of capital (K), the wage rate (w), and a desired level of output (Y), is:

$$L^* = \left(\frac{Y}{AK^{\alpha}}\right)^{\frac{1}{\beta}} \tag{24}$$

Cost minimization in the long run

In the long run, both capital (K) and labor (L) are variable. Given a Cobb-Douglas production function:

$$Y = AK^{\alpha}L^{\beta} \tag{25}$$

And the cost of production is determined by the wage rate (w), the rental rate of capital (r), and the amounts of labor and capital used:

$$C = wL + rK \tag{26}$$

We wish to minimize C subject to the production function. We set up a Lagrangian for this constrained optimization problem:

$$\mathcal{L} = wL + rK + \lambda \left(Y - AK^{\alpha}L^{\beta}\right) \tag{27}$$

The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial L} = w - \lambda \beta A K^{\alpha} L^{\beta-1} = 0 \tag{28}$$

$$\frac{\partial \mathcal{L}}{\partial K} = r - \lambda \alpha A K^{\alpha-1} L^{\beta} = 0 \tag{29}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = Y - A K^{\alpha} L^{\beta} = 0 \tag{30}$$

We can solve for λ from the first two equations and set them equal to each other, which yields the following condition:

$$\frac{w}{\beta A K^{\alpha} L^{\beta-1}} = \frac{r}{\alpha A K^{\alpha-1} L^{\beta}} \tag{31}$$

We can simplify this condition to:

$$\frac{K}{L} = \frac{\alpha}{\beta}\frac{w}{r} \tag{32}$$

We can express K as a function of L:

$$K = \frac{\alpha}{\beta}\frac{w}{r}L \tag{33}$$

And substitute this expression for K into the production function:

$$Y = A\left(\frac{\alpha}{\beta}\frac{w}{r}L\right)^{\alpha} L^{\beta} \tag{34}$$

Simplify to:

$$Y = A\left(\frac{\alpha}{\beta}\right)^{\alpha}\left(\frac{w}{r}\right)^{\alpha} L^{\alpha+\beta} \tag{35}$$

This equation represents the relationship between the output (Y) and the optimal amount of labor (L), given the wage rate (w), the rental rate of capital (r), the technology constant (A), and the output elasticities of capital (α) and labor (β).

To find the optimal amount of labor that minimizes cost in the long run, we can solve for L:

$$L = \left(\frac{Y}{A}\right)^{\frac{1}{\alpha+\beta}}\left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\left(\frac{r}{w}\right)^{\frac{\alpha}{\alpha+\beta}} \tag{36}$$

Substituting this expression for L into the optimal capital-labor ratio, we can find the optimal amount of capital (K):

$$K = \frac{\alpha}{\beta}\frac{w}{r}\left(\frac{Y}{A}\right)^{\frac{1}{\alpha+\beta}}\left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\left(\frac{r}{w}\right)^{\frac{\alpha}{\alpha+\beta}} \tag{37}$$

Simplifying gives:

$$K = \left(\frac{Y}{A}\right)^{\frac{1}{\alpha+\beta}}\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}\left(\frac{w}{r}\right)^{\frac{\beta}{\alpha+\beta}} \tag{38}$$